Network Game with Attacker and Protector Entities*

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Abstract. Consider an information network with harmful procedures called *attackers* (e.g., viruses); each attacker uses a probability distribution to choose a node of the network to damage. Opponent to the attackers is the *system protector* scanning and cleaning from attackers some part of the network (e.g., an edge or a path), which it chooses independently using another probability distribution. Each attacker wishes to maximize the probability of escaping its cleaning by the system protector; towards a conflicting objective, the system protector aims at maximizing the expected number of cleaned attackers.

We model this network scenario as a non-cooperative strategic game on graphs. We focus on the special case where the protector chooses a single edge. We are interested in the associated *Nash equilibria,* where no network entity can unilaterally improve its local objective. We obtain the following results:

- No instance of the game possesses a pure Nash equilibrium.
- Every mixed Nash equilibrium enjoys a graph-theoretic structure, which enables a (typically exponential) algorithm to compute it.
- We coin a natural subclass of mixed Nash equilibria, which we call *matching Nash equilibria,* for this game on graphs. Matching Nash equilibria are defined using structural parameters of graphs, such as independent sets and matchings.
  - We derive a characterization of graphs possessing matching Nash equilibria. The characterization enables a linear time algorithm to compute a matching Nash equilibrium on any such graph with a given independent set and vertex cover.
  - Bipartite graphs are shown to satisfy the characterization. So, using a polynomial-time algorithm to compute a perfect matching in a bipartite graph, we obtain, as our main result, an efficient graph-theoretic algorithm to compute a matching Nash equilibrium on any instance of the game with a bipartite graph.

* This work was partially supported by the IST Programs of the European Union under contract numbers IST-2001-33116 (FLAGS) and IST-2004-001907 (DELIS).

1 Introduction

Motivation and Framework. Consider an information network represented by an undirected graph. The nodes of the network are insecure and vulnerable to infection. A *system protector* (e.g., antivirus software) is available in the system; however, its capabilities are limited. The system protector can guarantee safety only to a small part of the network, such as a path or even a single edge, which it may choose using a probability distribution. A collection of *attackers* (e.g., viruses or Trojan horses) are also present in the network. Each attacker chooses (via a separate probability distribution) a node of the network; the node is harmed unless it is covered by the system protector. Apparently, the attackers and the system protector have conflicting objectives. The system protector seeks to protect the network as much as possible, while the attackers wish to avoid being caught by the network protector so that they be able to damage the network. Thus, the system protector seeks to maximize the expected number of attackers it catches, while each attacker seeks to maximize the probability it escapes from the system protector.

Naturally, we model this scenario as a strategic game with two kinds of players: the *vertex players* representing the attackers, and the *edge player* representing the system protector. The Individual Cost of each player is the quantity to be maximized by the corresponding entity. We are interested in the *Nash equilibria* [4, 5] associated with this game, where no player can unilaterally improve its Individual Cost by switching to a more advantageous probability distribution. We focus on the simplest case where the edge player chooses a singe edge.

Summary of Results. Our results are summarized as follows:

- We prove that no instance of the game has a pure Nash equilibrium (pure NE) (Theorem 1).
- We then proceed to study mixed Nash equilibria (mixed NE). We provide a graph-theoretic characterization of mixed NE (Theorem 2). Roughly speaking, the characterization yields that the support of the edge player and the vertex players are an edge cover and a vertex cover of the graph and an induced subgraph of the graph, respectively. Given the supports, the characterization provides a system of equalities and inequalities to be satisfied by the probabilities of the players. Unfortunately, this characterization only implies an exponential time algorithm for the general case.
- We introduce *matching* Nash equilibria, which are a natural subclass of mixed Nash equilibria with a graph-theoretic definition (Definition 1). Roughly speaking, the supports of vertex players in a matching Nash equilibrium form together an independent set of the graph, while each vertex in the supports of the vertex players is incident to only one edge from the support of the edge player.
- We provide a characterization of graphs admitting a *matching* Nash equilibrium (Theorem 3). We prove that a *matching* Nash equilibrium can be computed in linear time for any graph satisfying the characterization once a *suitable* independent set is given for the graph.

– We finally consider bipartite graphs for which we show that they satisfy the characterization of *matching* Nash equilibria; hence, they always have one (theorem 5). More importantly, we prove that a *matching* Nash equilibrium can be computed in polynomial time for bipartite graphs (6).

Due to space limits, some proofs are omitted; we include them in the full version of the paper [3].

Significance. Our work joins the booming area of *Algorithmic Game Theory.* Our work is the *first*[3] to model realistic scenarios about infected networks as a strategic game and study its associated Nash equilibria. Our results contribute towards answering the general question of Papadimitriou [6] about the complexity of Nash equilibria for our special game. Our results highlight a fruitful interaction between *Game Theory* and *Graph Theory.* We believe that our *matching* Nash equilibria (and extensions of them) will find further applications in other network games and establish themselves as a candidate Nash equilibrium for polynomial time computation in other settings as well.

2 Framework

Throughout, we consider an undirected graph $G(V, E)$, with $|V(G)| = n$ and $|E(G)| = m$. Given a set of vertices $X \subseteq V$, the graph $G \backslash X$ is obtained by removing from G all vertices of X and their incident edges. A graph H, is an *induced* subgraph of G, if $V(H) \subseteq V(G)$ and $(u, v) \in E(H)$, whenever $(u, v) \in E(G)$. Denote $\Delta(G)$ the maximum degree of the graph G. For any vertex $v \in V(G)$, denote $Neigh(v) = \{u : (u, v) \in E(G)\}$, the set of neighboring vertices of v. For a set of vertices $X \subseteq V$, denote $Neigh(X) = \{u \notin X : (u, v) \in E(G)$ for some $v \in X\}$. For all above properties of a graph G, when no confusion raises, we omit G.

2.1 The Model

An information network is represented as an undirected graph $G(V, E)$. The vertices represent the network hosts and the edges represent the communication links. We define a non-cooperative game $\Pi(G) = \langle \mathcal{N}, \{S_i\}_{i \in \mathcal{N}}, \{\mathsf{IC}\}_{i \in \mathcal{N}} \rangle$ as follows:

– The set of players is $\mathcal{N} = \mathcal{N}_{vp} \cup \mathcal{N}_{ep}$, where $\mathcal{N}_{vp}$ is a finite set of *vertex* players vp_i, $i \geq 1$, and $\mathcal{N}_{ep}$ is a singleton set of an *edge* player ep. Denote $\nu = |\mathcal{N}_{ep}|$.
– The strategy set S_i of each player vp_i, $i \in \mathcal{N}_{vp}$, is V; the strategy set S_{ep} of the player ep is E. Thus, the strategy set $\mathcal{S}$ of the game is $\left(\underset{i \in N_{vp}}{\times} S_i \right) \times S_{ep} = V^{|\mathcal{N}_{vp}|} \times E$.

[3] To the best of our knowledge, [1] is a single exception. It considers inoculation strategies for victims of viruses and establishes connections with variants of the Graph Partition problem.

– Take any *strategy profile* $\mathbf{s} = \langle s_1, \ldots, s_{|\mathcal{N}_{vp}|}, s_{ep} \rangle \in \mathcal{S}$, also called a *configuration.*

  - The *Individual Cost* of vertex player vp_i is a function $\mathsf{IC}_i : \mathcal{S} \to \{0, 1\}$ such that $\mathsf{IC}_i(\mathbf{s}) = \begin{cases} 0, & s_i \in s_{ep} \\ 1, & s_i \notin s_{ep} \end{cases}$; intuitively, vp_i receives 1 if it is not caught by the edge player, and 0 otherwise.
  - The *Individual Cost* of the edge player ep is a function $\mathsf{IC}_{ep} : \mathcal{S} \to \mathbb{N}$ such that $\mathsf{IC}_{ep}(\mathbf{s}) = |\{s_i : s_i \in s_{ep}\}|$.

The configuration $\mathbf{s}$ is a *pure Nash equilibrium* [4, 5] (abbreviated as *pure NE*) if for each player $i \in \mathcal{N}$, it minimizes IC_i over all configurations $\mathbf{t}$ that differ from $\mathbf{s}$ only with respect to the strategy of player i.

A *mixed strategy* for player $i \in \mathcal{N}$ is a probability distribution over its strategy set S_i; thus, a mixed strategy for a vertex player (resp., edge player) is a probability distribution over vertices (resp., over edges) of G. A *mixed strategy profile* $\mathbf{s}$ is a collection of mixed strategies, one for each player. Denote $P_{\mathbf{s}}(ep, e)$ the probability that edge player ep chooses edge $e \in E(G)$ in $\mathbf{s}$; denote $P_{\mathbf{s}}(vp_i, v)$ the probability that vertex player vp_i chooses vertex $v \in V$ in $\mathbf{s}$. Note $\sum_{v \in V} P_{\mathbf{s}}(vp_i, v) = 1$ for each vertex player vp_i; similarly, $\sum_{e \in E} P_{\mathbf{s}}(ep, e) = 1$. Denote $P_{\mathbf{s}}(vp, v) = \sum_{i \in \mathcal{N}_{vp}} P_{\mathbf{s}}(vp_i, v)$ the probability that vertex v is chosen by some vertex player in $\mathbf{s}$.

The *support* of a player $i \in \mathcal{N}$ in the configuration $\mathbf{s}$, denoted $D_{\mathbf{s}}(i)$, is the set of pure strategies in its strategy set to which i assigns strictly positive probability in $\mathbf{s}$. Denote $D_{\mathbf{s}}(vp) = \bigcup_{i \in \mathcal{N}_{vp}} D_{\mathbf{s}}(i)$; so, $D_{\mathbf{s}}(vp)$ contains all pure strategies (that is, vertices) to which some vertex player assigns strictly positive probability. Let also $ENeigh_{\mathbf{s}}(v) = \{(u, v) E : \ (u, v) \in D_{\mathbf{s}}(ep)\}$; that is $ENeigh_{\mathbf{s}}(v)$ contains all edges incident to v that are included in the support of the edge player in $\mathbf{s}$.

A mixed strategic profile $\mathbf{s}$ induces an *Expected Individual Cost* IC_i for each player $i \in \mathcal{N}$, which is the expectation, according to $\mathbf{s}$, of its corresponding Individual Cost (defined previously for pure strategy profiles). The mixed strategy profile $\mathbf{s}$ is a *mixed Nash equilibrium* [4, 5] (abbreviated as mixed NE) if for each player $i \in \mathcal{N}$, it maximizes IC_i over all configurations $\mathbf{t}$ that differ from $\mathbf{s}$ only with respect to the mixed strategy of player i.

For the rest of this section, fix a mixed strategy profile $\mathbf{s}$. For each vertex $v \in V$, denote $Hit(v)$ the event that the edge player hits vertex v. So, the probability (according to $\mathbf{s}$) of $Hit(v)$ is $P_{\mathbf{s}}(Hit(v)) = \sum_{e \in ENeigh_{\mathbf{s}}(v)} P_{\mathbf{s}}(ep, e)$. Define the minimum hitting probability $P_{\mathbf{s}}$ as $\min_v P_{\mathbf{s}}(Hit(v))$. For each vertex $v \in V$, denote $m_{\mathbf{s}}(v)$ the expected number of vertex players choosing v (according to $\mathbf{s}$). For each edge $e = (u, v) \in E$, denote $m_{\mathbf{s}}(e)$ the expected number of vertex players choosing either u or v; so, $m_{\mathbf{s}}(e) = m_{\mathbf{s}}(u) + m_{\mathbf{s}}(v)$. It is easy to see that for each vertex $v \in V$, $m_{\mathbf{s}}(v) = \sum_{i \in \mathcal{N}_{vp}} P_{\mathbf{s}}(vp_i, v)$. Define the maximum expected number of vertex players choosing e in $\mathbf{s}$ as $\max_e m_{\mathbf{s}}(e)$. We proceed to calculate the Expected Individual Cost. Clearly, for the vertex player $vp_i \in \mathcal{N}_{vp}$,

$$\begin{aligned}\mathsf{IC}_i(\mathbf{s}) &= \sum_{v \in V(G)} P_{\mathbf{s}}(vp_i, v) \cdot (1 - P_{\mathbf{s}}(Hit(v)) \\ &= \sum_{v \in V(G)} P_{\mathbf{s}}(vp_i, v) \cdot (1 - \sum_{e \in ENeigh_{\mathbf{s}}(v)} P_{\mathbf{s}}(ep, e))\end{aligned} \tag{1}$$

For the edge player ep,

$$\begin{aligned}\mathsf{IC}_{ep}(\mathbf{s}) &= \sum_{e=(u,v) \in E(G)} P_{\mathbf{s}}(ep, e) \cdot (m_{\mathbf{s}}(u) + m_{\mathbf{s}}(v)) \\ &= \sum_{e=(u,v) \in E(G)} P_{\mathbf{s}}(ep, e) \cdot (\sum_{i \in N_{vp}} P_{\mathbf{s}}(vp_i, u) + P_{\mathbf{s}}(v_i, v))\end{aligned} \tag{2}$$

2.2 Background from Graph Theory

Throughout this section, we consider the (undirected) graph $G = G(V, E)$. $G(V, E)$ is *bipartite* if its vertex set V can be partitioned as $V = V_1 \cup V_2$ such that each edge $e = (u, v) \in E$ has one of its vertices in V_1 and the other in V_2. Such a graph is often referred to as a V_1, V_2-bigraph. Fix a set of vertices $S \subseteq V$. The graph G is an S-*expander* if for every set $X \subseteq S$, $|X| \leq |Neigh_G(X)|$.

A set $M \subseteq E$ is a *matching* of G if no two edges in M share a vertex. Given a matching M, say that set $S \subseteq V$ is *matched in* M if for every vertex $v \in S$, there is an edge $(v, u) \in\in M$. A *vertex cover* of G is a set $V' \subseteq V$ such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$. An *edge cover* of G is a set $E' \subseteq E$ such that for every vertex $v \in V$, there is an edge $(v, u) \in E'$. Say that an edge $(u, v) \in E$ (resp., a vertex $v \in V$) is *covered* by the vertex cover V' (resp., the edge cover E') if either $u \in V'$ or $v \in V'$ (resp., if there is an edge $(u, v) \in E'$). A set $IS \subseteq V$ is an *independent set* of G if for all vertices $u, v \in IS$, $(u, v) \notin E$. Clearly, $IS \subseteq V$ is an independent set of G if and only if the set $VC = V \backslash IS$ is a vertex cover of G. We will use the following consequence of Hall's Theorem [2, Chapter 6] on the marriage problem.

Proposition 1 (Hall's Theorem). *A graph G has a matching M in which the vertex set $X \subseteq V$ is matched if and only if for each for each subset $S \subseteq X$, $|N(S)| \geq |S|$.*

3 Nash Equilibria

Theorem 1. *If G contains more than one edges, then $\Pi(G)$ has no pure Nash equilibrium.*

Proof. Consider any graph G with at least two edges and any configuration $\mathbf{s}$ of $\Pi(G)$. Let e the edge selected by the edge player in $\mathbf{s}$. Since G contains more than one edges, there exists an $e' \in E$ not selected by the edge player in $\mathbf{s}$, such that e and e' contain at least one different endpoint, assume u. If there is at least one v.p. located on e, it will prefer to go to u so that not to get arrested by the edge player and gain more. Thus, this case can not be a pure NE for the vertex players. Otherwise, the edge e contains no vertex player. But in this case, the edge player would like to change current action and select another edge, where there is at least one vertex player, so that to gain more. Thus, again this case can not be a pure NE, for the edge player this time. Since always in any case, one of the two kinds of players is not satisfied by $\mathbf{s}$, $\mathbf{s}$ is not a pure NE. □

Theorem 2 (Characterization of Mixed NE). *A mixed strategy profile* $\mathbf{s}$ *is a Nash equilibrium for any* $\Pi(G)$ *if and only if:*

1. $D_{\mathbf{s}}(ep)$ *is an* edge cover *of* G *and* $D_{\mathbf{s}}(vp)$ *is a* vertex cover *of the graph obtained by* $D_{\mathbf{s}}(ep)$.
2. *The probability distribution of the edge player over* E, *is such that, (a)* $P_{\mathbf{s}}(Hit(v)) = P_{\mathbf{s}}(Hit(u)) = \min_v P_{\mathbf{s}}(Hit(v)),\ \forall\ u,v \in D_{\mathbf{s}}(vp)$ *and (b)* $\sum_{e\in D_{\mathbf{s}}(ep)} P_{\mathbf{s}}(ep,e) = 1$.
3. *The probability distributions of the vertex players over* V *are such that, (a)* $m_{\mathbf{s}}(e_1) = m_{\mathbf{s}}(e_2) = \max_e m_{\mathbf{s}}(e),\ \ \forall\ e_1 = (u_1,v_1), e_2 = (u_2,v_2) \in D_{\mathbf{s}}(ep)$ *and (b)* $\sum_{v\in V(D_{\mathbf{s}}(ep))} m_{\mathbf{s}}(v) = \nu$.

4 Matching Nash Equilibria

The obvious difficulty of solving the system of Theorem 2 directs us in trying to investigate the existence of some polynomially computable, solutions of the system, corresponding to mixed NE of the game. We introduce a class of such configurations, called *mathing*. We prove that they can lead to mixed NE, we investigate their existence and their polynomial time computation.

Definition 1. *A* **matching configuration** $\mathbf{s}$ *of* $\Pi(G)$ *satisfies:* **(1)** $D_{\mathbf{s}}(vp)$ *is an independent set of* G *and* **(2)** *each vertex* v *of* $D_{\mathbf{s}}(vp)$ *is incident to only one edge of* $D_{\mathbf{s}}(ep)$.

Lemma 1. *For any graph* G, *if in* $\Pi_{\mathsf{E}}(G)$ *there exists a* matching *configuration which additionally satisfies condition* **1** *of Theorem 2, then there exists probability distributions for the vertex players and the edge player such that the resulting configuration is a mixed Nash equilibrium for* $\Pi_{\mathsf{E}}(G)$. *These distributions can be computed in polynomial time.*

Proof. Consider any configuration $\mathbf{s}$ as stated by the lemma (assuming that there exists one) and the following probability distributions of the vertex players and the edge player on $\mathbf{s}$:

$$\forall e \in D_{\mathbf{s}}(ep),\ P_{\mathbf{s}}(ep,e) := 1/|D_{\mathbf{s}}(ep)|,\ \forall e' \in E,\ e' \notin D_{\mathbf{s}}(ep),\ P_{\mathbf{s}}(ep,e') := 0 \tag{3}$$

$$\begin{array}{l} \forall\ i \in \mathcal{N}_{vp},\ \forall\ v \in D_{\mathbf{s}}(vp),\ P_{\mathbf{s}}(vp_i,v) := \frac{1}{|D_{\mathbf{s}}(vp)|}, \\ \forall u \in V,\ u \notin D_{\mathbf{s}}(vp),\ P_{\mathbf{s}}(vp_i,u) := 0 \end{array} \tag{4}$$

Proposition 2.

$$\forall\ v \in D_{\mathbf{s}}(vp),\ \ m_{\mathbf{s}}(v) = \frac{\nu}{|D_{\mathbf{s}}(vp)|}\ \ \textit{and}\ \ \forall\ u \in V,\ u \notin D_{\mathbf{s}}(vp),\ m_{\mathbf{s}}(u) = 0$$

We show that $\mathbf{s}$ satisfies all conditions of Theorem 2, thus it is a mixed NE. **2.(a)**: $P_{\mathbf{s}}(Hit(v)) = \frac{1}{|D_{\mathbf{s}}(ep)|},\ \forall\ \ v \in D_{\mathbf{s}}(vp)$, by condition (2) of the definition of a *matching* configuration and equation (3) above. **3.(a)**: $m_{\mathbf{s}}(e_1) = m_{\mathbf{s}}(v_1) + m_{\mathbf{s}}(u_1) = 0 + \frac{\nu}{|D_{\mathbf{s}}(vp)|} = \frac{\nu}{|D_{\mathbf{s}}(vp)|},\ \forall\ e_1 = (u_1, v_1) \in D_{\mathbf{s}}(ep)$, because D_{ep} is an edge cover of G (by assumption), D_{vp} is an independent set of G (condition (1) of the definition of a *matching* configuration) and recalling Proposition 2 above. **3.(c)**: Since $D_{evp}(\mathbf{s})$ is an edge cover of G (by assumption) and by Proposition 2, we have $\sum_{v \in V(D_{\mathbf{s}}(ep))} m_{\mathbf{s}}(v) = \sum_{v \in V} \frac{\nu}{|D_{\mathbf{s}}(vp)|} = |D_{\mathbf{s}}(vp)| \cdot \frac{\nu}{|D_{\mathbf{s}}(vp)|} = \nu$. The rest of the conditions, can be easily shown to be fulfilled in $\mathbf{s}$; see [3]. $\square$

Definition 2. *A* matching *configuration which additionally satisfies condition* **1** *of Theorem 2 is called a* **matching mixed NE**.

We proceed to characterize graphs that admit *matching* Nash equilibria.

Theorem 3. *For any graph G, $\Pi(G)$ contains a* matching *mixed Nash equilibrium if and only if the vertices of the graph G can be partitioned into two sets IS, VC ($VC \cup IS = V$ and $VC \cap IS = \emptyset$), such that IS is an independent set of G (equivalently, VC is a vertex cover of the graph) and G is a VC-expander graph.*

Proof. We first prove that if G has an independent set IS and the graph G is a VC-expander graph, where $VC = V \backslash IS$, then $\Pi_{\mathsf{E}}(G)$ contains a matching *mixed NE.* By the definition of a VC-expander graph, it holds that $Neigh(VC') \geq VC'$, for all $VC' \subseteq VC$. Thus, by the Marriage's Theorem 1, G has a matching M such that each vertex $u \in VC$ is matched into $V \backslash VC$ in M; that is there exists an edge $e = (u, v) \in M$, where $v \in V \backslash VC = IS$. Partition IS into two sets IS_1, IS_2, where set IS_1 consists of vertices $v \in IS$ for which there exists an $e = (u, v) \in M$ and $u \in VC$. Let IS_2 the remaining vertices of the set, i.e. $IS_2 = \{v \in IS : \forall\ u \in VC,\ (u, v) \notin M\}$.

Now, recall that there is no edge between any two vertices of set IS, since it is independent set, by assumption. Henceforth, since G is a connected graph, $\forall\ u \in IS_2 \subseteq IS$, there exists $e = (u, v) \in E$ and moreover $v \in V \backslash IS = VC$. Now, construct set $M_1 \subseteq E$ consisting of all those edges. That is, initially set $M := \emptyset$ and then for each $v \in IS_2$, add one edge $(u, v) \in E$ in M_1. Note that, by the construction of the set M_1, each edge of it is incident to only one vertex of IS_2. Next, construct the following configuration $\mathbf{s}$ of $\Pi_{\mathsf{E}}(G)$: Set $D_{\mathbf{s}}(vp) := IS$ and $D_{\mathbf{s}}(ep) := M \cup M_1$.

We first show that that $\mathbf{s}$ is a *matching* configuration. Condition (1) of a matching configuration is fulfilled because $D_{\mathbf{s}}(vp)(= IS)$ is an independent set. We show that condition (2) of a *matching* configuration is fulfilled. Each vertex of set IS belongs either to IS_1 or to IS_2. By definition, each vertex of IS_1 is incident to only one edge of M and each vertex of IS_2 is incident to no edge in M. Moreover, by the construction of set M_1, each vertex of IS_2 is incident to exactly one edge of M_1. Thus, each vertex $v \in D_{\mathbf{s}}(vp)(= IS)$ is incident to only one edge of $D_{\mathbf{s}}(ep)(= M \cup M_1)$, i.e. condition (2) holds as well. Henceforth, $\mathbf{s}$ is a *matching* configuration.

We next show that condition **1** of Theorem 2 is satisfied by $\mathbf{s}$. We first show that $D_{\mathbf{s}}(ep)$ is an edge cover of G. This is true because (i) set $M \subseteq D_{\mathbf{s}}(ep)$ covers all vertices of set VC and IS_1, by its construction and (ii) set $M_1 \subseteq D_{\mathbf{s}}(ep)$ covers all vertices of set IS_2, which are the remaining vertices of G not covered by set M, also by its construction. We next show that $D_{\mathbf{s}}(vp)$ is a vertex cover of the subgraph of G obtained by set $D_{\mathbf{s}}(ep)$. By the definition of sets $IS_1,\ IS_2 \subseteq IS$, any edge $e \in M$ is covered by a vertex of set IS_1 and each edge $e \in M_1$ is covered by a vertex of set IS_2. Since $D_{\mathbf{s}}(ep) = M \cup M_1$, we get that all edges of the set are covered by $D_{\mathbf{s}}(vp) = IS_1 \cup IS_2$. This result combined with the above observation on $D_{\mathbf{s}}(ep)$ concludes that condition **1** of Theorem 2 is satisfied by $\mathbf{s}$. Henceforth, by lemma 1, it can lead to a *matching* mixed NE of $\Pi_{\mathsf{E}}(G)$.

We proceed to show that if G contains a matching *mixed NE, assume* $\mathbf{s}$*, then G has an independent set IS and the graph G is a VC-expander graph, where $VC = V \backslash IS$.* Define sets $IS = D_{\mathbf{s}}(vp)$ and $VC = V \backslash IS$. We show that these sets satisfy the above requirements for G. Note first that, set IS is an independent of G since $D_{\mathbf{s}}(vp)$ is an independent set of G by condition (1) of the definition of a *matching* configuration.

We next show G contains a matching M such that each vertex of VC is matched into $V \backslash VC$ in M. Since $D_{\mathbf{s}}(ep)$ is an edge cover of G (condition **1** of a mixed NE of Theorem 2), for each $v \in VC$, there exists an edge $(u, v) \in D_{\mathbf{s}}(ep)$. Note that for edge (u, v), it holds that $v \in IS$, since otherwise IS would not be a vertex cover of $D_{\mathbf{s}}(ep)$ (Condition **1** of a mixed NE). Now, construct a set $M \subseteq E$ consisting of all those edges. That is, That is, initially set $M := \emptyset$ and then for each $v \in VC$, add one edge $(u, v) \in D_{\mathbf{s}}(ep)$ in M. By the construction of set M and condition (2) of a *matching* mixed NE, we get that M is a matching of G and that each vertex of VC is matched into $V \backslash VC$ in M. Thus, by the Marriage's Theorem 1, we get that $Neigh(VC') \geq VC'$, for all $VC' \subseteq VC$ and so G is a VC-expander and condition (2) of a matching configuration also holds in $\mathbf{s}$. $\square$

4.1 A Polynomial Time Algorithm

The previous Theorems and Lemmas enables us to develop a polynomial time algorithm for finding *matching* mixed NE for any $\Pi(G)$, where G is a graph satisfying the requirements of Theorem 3. The Algorithm is described in pseudocode in Figure 1.

Theorem 4. *Algorithm A computes a* matching *mixed Nash equilibrium for $\Pi(G)$ and needs linear time $O(m)$.*

Algorithm $A(\Pi(G), IS, VC)$

INPUT: A game $\Pi(G)$ and a partition of $V(G)$ into sets IS, $VC = V \backslash IS$, such that IS is an independent set of G and G is a VC-expander graph.
OUTPUT: A mixed NE $\mathbf{s}$ for $\Pi(G)$.

1. Compute a set $M \subseteq E$, as follows:
   (a) *Initialization*: Set $M := \emptyset$, $Matched := \emptyset$ (currently matched vertices in M), $Unmatched := VC$ (currently unmatched vertices in M vertices of VC), $Unused := IS$, $i := 1$, $G_i := G$ and $M_1 := \emptyset$.
   (b) While $Unmatched \neq \emptyset$ Do:
      i. Consider a $u \in Unmatched$.
      ii. Find a $v \in Unused$ such that $(u, v) \in E_i$. Set $M := M \cup (u, v)$, $Unused := Unused \backslash \{v\}$.
      iii. *Prepare next iteration*: Set $i := i + 1$, $Matched := Matched \cup \{u\}$, $Unmatched := Unmatched \backslash \{u\}$, $G_i := G_{i-1} \backslash u \backslash v$.
2. Partition set IS into two sets IS_1, IS_2 as follows: $IS_1 := \{u \in IS : \exists\, (u, v) \in M\}$ and $IS_2 := IS \backslash IS_1$. Note that $IS_2 := \{u \in IS : \nexists\, (u, v) \in M,\ v \in VC\}$.
   Compute a set M_1 as follows: $\forall\ u \in IS_2$, set $M_1 := M_1 \cup (u, v)$, for any $(u, v) \in E$, $v \in VC$.
3. Define a s with the following support: $D_{\mathbf{s}}(vp) := IS$, $D_{\mathbf{s}}(ep) := M \cup M_1$.
4. Determine the probabilities distributions of the vertex players and the edge player of configuration s' using equations (3) and (4) of Lemma 1.

Fig. 1. Algorithm A.

5 Bipartite Graphs

Lemma 2. *In any bipartite graph G there exists a matching M and a vertex cover VC such that* **(1)** *every edge in M contains exactly one vertex of VC and* **(2)** *every vertex in VC is contained in exactly one edge of M.*

Proof. Let X, Y the bipartition of the bipartite graph G. Consider any minimum vertex cover of the graph G, VC. We are going to construct a matching M of G so that conditions (1) and (2) of the Lemma hold. Let R the vertices of VC contained in set X, i.e. $R = VC \cap X$ and T the vertices of VC contained in set Y, i.e. $T = VC \cap Y$. Note that $VC = R \cup T$. Let H and H' the subgraphs of G induced by $R \cup (Y - T)$ and $T \cup (X - R)$, respectively. We are going to show that G contains a matching in M as required by the Lemma.

Since $R \cup T$ is a vertex cover, G has no edge from $Y - T$ to $X - R$. We show that for each $S \subseteq R$, $N_H(S) \subseteq Y - T$. If $|N_H(S)| < |S|$, then we can substitute $N_H(S)$ for S in VC to obtain a smallest vertex cover $(*1)$. This is true because (i) $N_H(S)$ covers all edges incident to S that are not covered by T and (ii) since G is a bipartite graph there are no edges between the vertices of set S, so that a possible substitute of set S do not need to cover any such edge.

Thus, $|N_H(S)| \geq |S|$, for all $S \subseteq R$. By Hall's Theorem (Theorem 1), H has a matching M_H such that each vertex of R is matched in M_H. Using similar arguments for set T, we can prove that for each $S' \subseteq T$, $|N_{H'}(S')| \geq |S'|$. Henceforth, H' has a matching $M_{H'}$ such that each vertex of T is matched in $M_{H'}$. Now define $M = M_H \cup M_{H'}$. Since each H, H' is an induced subgraph of G and the two subgraphs have disjoint sets of vertices, we get that M is matching of G and that each vertex of $VC = R \cup T$ is matched in M. This result combined with the fact that M is a matching of G concludes that condition (2) of the Lemma holds.

We proceed to prove condition (1). That is, to show that every edge of M contains exactly one vertex of VC. Observe first that by the construction of set M, every edge of M contains at least one vertex of VC. Moreover, note that only one of the endpoints of the edge is contained in M. This is true because by the construction of set M each edge of set M matches either (i) a vertex of set $R \subseteq X$ to a vertex of set $(Y - T) \subseteq Y$ or (ii) a vertex of set $T \subseteq Y$ to a vertex of set $(X - R) \subseteq X$. So, for any case exactly one of the two endpoints of the edge is not contained in VC. $\square$

Lemma 3. *Any X, Y-bigraph graph G can be partitioned into two sets IS, VC ($IS \cup VC = V$ and $IS \cap VC = \emptyset$) such that VC is a vertex cover of G (equivalently, IS is an independent set of G) and G is a VC-expander graph.*

Lemma 3 and Theorem 3 finally imply:

Theorem 5. *Any $\Pi(G)$ for which G is a connected bipartite graph, contains a* matching *mixed Nash equilibrium.*

Finally, we proved that,

Theorem 6. *For any $\Pi(G)$, for which G is a bipartite graph, a* matching *mixed Nash equilibrium of $\Pi(G)$ can be computed in polynomial time,* $\max\{O(m\sqrt{n}), O(n^{2.5}/\sqrt{\log n})\}$*, using Algorithm A.*

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